XLVIII. *Concise Rules for computing the Effects of Refraction and Parallax in varying the Apparent Distance of the* Moon *from the* Sun *or a* Star; *also an easy Rule of Approximation for computing the Distance of the* Moon *from a* Star, *the Longitudes and Latitudes of both being given, with Demonstrations of the same : By the Rev.* Nevil Maskelyne, *A. M. Fellow of* Trinity College, *in the University of* Cambridge, *and F. R. S.*

Read Nov. 15, 1764.

THE following rules, excepting one, are the same which I have already communicated to the Royal Society, but without demonstration, in a letter to the reverend Dr. Birch from St. Helena, containing the results of my observations of the distance of the Moon from the Sun and fixed stars, taken in my voyage thither, for finding the longitude of the ship from time to time; since printed in Part II. Vol. LII. of the Philosophical Transactions for 1762. The two rules for the correction of refraction and parallax I have also since communicated to the public in my British Mariner's Guide to the discovery of longitude from like observations of the Moon; and have added in the Preface a rule for computing a second but smaller correction of parrallax, necessary on account of a small imperfection

imperfection lying in the firft rule derived from the fluxions of a fpherical triangle. To the rules I have here fubjoined their demonftrations.

With refpect to the ufefulnefs of thefe rules, I cannot but entertain hopes that they will appear more fimple and eafy than any yet propofed for the fame purpofe: the laft rule, for computing the diftance of the Moon from a ftar, though only an approximation, being fo very exact, feems particularly adapted for the conftruction of a nautical Ephemeris, containing the diftances of the Moon from the Sun and proper fixed ftars ready calculated for the purpofe of finding the longitude from obfervations of the Moon at fea; an affiftance which, in an age abounding with fo many able computers, mariners need not doubt they will be provided with, as foon as they manifeft a proper difpofition to make ufe of it.

A R U L E

To compute the contraction of the apparent diftance of any two heavenly bodies by refraction; the zenith diftances of both, and their diftance from each other being given nearly.

Add together the tangents of half the fum, and half the difference of the zenith diftances; their fum, abating 10 from the index, is the tangent of arc the firft. To the tangent of arc the firft, juft found, add the co-tangent of half the diftance of the ftars; the fum, abating 10 from the index, is the tangent of arc the fecond. Then add together the tangent of double the firft arc, the co-fecant of double the

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.

Fig. 6.

Fig. 7.
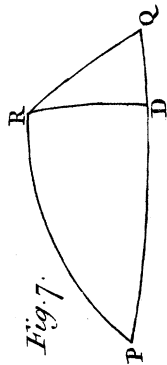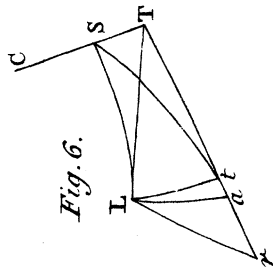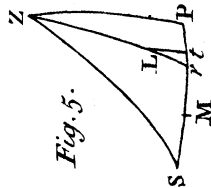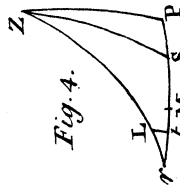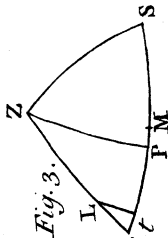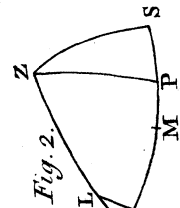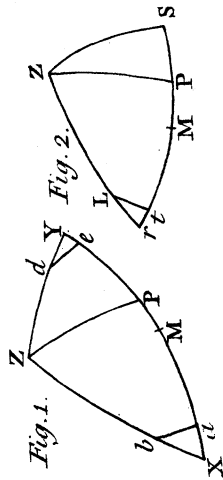
double the fecond arch, and the conftant logarithm of 114″ or 2,0569 : the fum abating 20 from the index, is the logarithm of the number of feconds required, by which the diftance of the ftars is contracted by refraction : which therefore added to the obferved diftance gives the true diftance cleared from the effect of refraction.

Explication of the foundation of the preceding rule.

This rule is founded upon an hypothefis that the refraction in altitude is as the tangent of the zenith diftance : and the refraction at the altitude of 45 degrees being 57″, according to Dr. Bradley's obfervations, therefore the refraction at any altitude, calling the radius unity is $=$ 57″ $\times$ tangent of the zenith diftance. This rule is exact enough for the purpofe of the calculation of the longitude from obfervations of the diftance of the Moon from ftars at fea as low down as the altitude of 10°, for there the error is only 10″ from the truth. But, if the altitude of the Moon or ftar be lefs than 10°, the rule may be ftill made to anfwer fufficiently, by only firft correcting the obferved zenith diftances by fubftracting from them three times the refraction correfponding to them taken out of any common table of refraction, and making the computation with the zenith diftances thus corrected. This correction depends upon Dr. Bradley's rule for refraction, which he found to anfwer, in a manner exactly, from the zenith quite down to the horifon, namely that the refraction is $=$ 57″ $\times$ tangent of the apparent zenith diftance

M m leffened

leſſened by three times the correſponding refraction taken out of any common table.

Demonſtration of the preceding rule.

Let Z X Y Tab. XVII. Fig. 1. repreſent a ſpherical triangle formed by great circles joining the zenith Z and the ſtars X and Y. Refraction acting in the vertical circles Z X and Z Y will carry the ſtar X nearer the zenith by a quantity $X b = 57'' \times$ tangent of Z X, and the ſtars Y towards Z by the quantity $Y d = 57'' \times$ tangent of Z Y; ſo that the apparent diſtance of the two ſtars will be $b d$ inſtead of X Y or leſs than X Y, the true diſtance, by the Sun of the two little ſpaces X a, Y e, terminated by the perpendiculars $b a$ and $d e$. The little ſpace $X a = X b \times$ coſine of the angle Z X Y (calling radius unity) $= 57'' \times$ tang. of Z X $\times$ coſine of angle Z X Y, or, by ſpherics, $= 57'' \times$ tang. of X P. (Z P being an arch drawn from Z perpendicular to the arch X Y). In like manner the little ſpace $Y e = 57'' \times$ tang. of Y P; and therefore $X a + Y e$ or the total effect of refraction $= 57'' \times \overline{\text{tang. X P} + \text{tang Y P}}$. Let M be the middle of the arch X Y, and put the tangent of X M or Y M or $\frac{1}{2}$ X Y $= t$, and the tangent of M P, or the diſtance of the perpendicular Z P from M the middle of the arch X Y $= n$. By trigonometry, tang. of X P or $X M + M P = \frac{t+n}{1-tn}$, and tang. of Y P or $Y M - M P = \frac{t-n}{1+tn}$; the ſum of which, or tang. X P $+$ tang. Y P $= \frac{t+n}{1-tn} + \frac{t-n}{1+tn} = \frac{2t+2tn^2}{1-t^2n^2} = \frac{2tn}{1-t^2n^2} \times \frac{1+n^2}{2n} \times 2$. Now

$2tn$

$\frac{2\,t\,n}{1-t^2\,n^2}$ is the tangent of double the angle whofe tangent is $t\,n$, and $t\,n$ or the product of the tangents of $\frac{1}{2}$ X Y and M P, by fpherics, is equal to the product of the tangents of half the fum and half the difference of the zenith diftances Z X and Z Y; whence $\frac{2\,t\,n}{1-t^2\,n^2}$ is equal to the tangent of double arch the firft found by the rule. Alfo arch the fecond found by the rule being by fpherics $=$ M P, whofe tangent is reprefented by n, and $\frac{1+n^2}{2\,n}$ being by trigonometry equal to the cofecant of double the arch whofe tangent is n, therefore $\frac{1+n^2}{2\,n}=$ cofecant of 2 M P or double arch the fecond. Whence the rule is manifeft; namely, that X a + Y e, the total effect of refraction in contracting the apparent diftance of the two ftars $= 57''\times \overline{\text{tan. X P}+\text{tan. Y P}}$ $=$ double $57''$ or $114'' \times$ tang. of double the firft arch $\times$ cofecant of double the fecond arch. Q. E. D.

R E M A R K.

When the perpendicular arch Z P, let fall from the Zenith on the arch X Y, falls without the triangle Z X Y, the effect of refraction in diminifhing the apparent diftance of the ftars X, Y is the difference of X a and Y e: but the rule being general, gives always the fum or difference, which-ever it be, which is a great advantage, and removes all grounds of ambiguity in the correction of refraction; as the the total effect thereof always diminifhes the diftance of two ftars from each other, however they are pofited.

A

A RULE

To compute the contraction or augmentation of the apparent diftance of the Moon from a ftar, on account of the Moon's parallax ; the zenith diftances of the Moon and ftar, and their diftance from each other being given nearly.

Add together the tangents of half the fum, and half the difference of the zenith diftances of the Moon and ftar, and the cotangent of half the diftance of the Moon from the ftar; the fum, abating 20 from the index, is the tangent of an arch, which call A. Then, if the zenith diftance of the Moon is greater than that of the ftar, take the Sum of the arch A, juft found, and half the diftance of the Moon from the ftar; but, if the zenith diftance of the Moon be lefs than that of the ftar, take the difference of the faid arch A and half the diftance of the Moon from the ftar ; and the fum, or difference call B. To the tangent of B, thus found, add the cofine of the Moon's zenith diftance, and the logarithm of the Moon's horizontal parallax, exprefled in minutes and decimals; the fum, abating 20 from the index, is the logarithm of the effect of parallax, tending always to augment the apparent diftance of the Moon from the ftar ; except the zenith diftance of the Moon be lefs than that of the ftar, and, at the fame time, the arch A be greater than half the diftance of the Moon from the ftar, in which cafe the effect of parallax diminifhes the apparent diftance of the Moon from the ftar.

2   DEMON-

DEMONSTRATION.

In the fpherical triangle Z L S, fee Fig. 2, 3, 4, and 5th, Z reprefents the zenith, L the Moon, and S the ftar; the effect of parallax depreffing the Moon from L to r, r is the apparent place of the Moon, and r S the apparent diftance of the Moon from the ftar; let fall the perpendicular L t upon r S, produced if neceffary, and $r t$ will be the difference of L S and r S, or the effect of parallax. Draw the arch Z P perpendicular to to r S, and let M be the middle of r S. The Moon's parallax in altitude, being to her horizontal parallax, as the fine of her apparent zenith diftance, to the radius, L r = Moon's horizontal parallax $\times$ fine of Z r; and $r t$ the effect of parallax upon the apparent diftance of the Moon from the ftar will be = L r $\times$ cof. Z r S = horizontal parallax $\times$ fin. Z r $\times$ cof. Z r S (or, becaufe tan. r P : cof. Z r P :: tan Z r : rad :: fin Z r : cos Z r; and therefore fin. Z r $\times$ cos Z r S = cof. Z r $\times$ tan r P) = horizontal parallax $\times$ cof. Z r $\times$ tan. r P agreeably to the rule. For it is evident by fpherics that the arch A, found by the rule, is the fame with M P the diftance of the perpendicular from the middle of the arch r S : and it is evident, by the infpection of the figures, that the arch B or r P is equal to the fum of r M and M P, if the zenith diftance of the Moon be greater than that of the ftar, as in Fig. 2d and 4th; but is the difference of r M and M P, if the zenith diftance of the Moon be lefs than that of the ftar, as in Fig. 3d and 5th. Laftly, it may appear from the

consideration

confideration of the figures, that, as the effect of par-
allax depreffes the Moon directly towards the hori-
zon, fo it will always encreafe her apparent diftance
from a ftar, except in the cafe reprefented by Fig.
5th ; that is to fay, unlefs the zenith diftance of the
Moon be lefs than that of the ftar, and, at the fame
time, the arch M P be greater than *r* M or half
the diftance of the Moon from the ftar. Q. E. D.

Remarks on the ufe of the two foregoing rules.

It has been remarked, after the rule for refraction
above, that if the altitudes of the Moon or ftar are
under 10 degrees, the zenith diftances muft be firft
leffened by 3 times the refractions correfponding to
their refpective altitudes before the effect of refrac-
tion be computed.

But in order to compute the effect of parallax from
the fecond rule, the obferved diftance of the Moon
from the ftar muft be firft corrected by adding the
effect of refraction to it found by rule the firft ;
as muft the obferved altitudes of the Moon and ftar
be alfo corrected by taking from them their refpec-
tive refraction in altitude, and the corrected arches
thus found muft be made ufe of in computing the
parallax. Only, if the altitudes of the Moon and
ftar are both 10 degrees or more, part of the calcu-
lation of rule the fecond may be faved, and arch the
fecond, found by rule the firft, taken for arch A in
the fecond rule without any fenfible error. In this
cafe, it will be moft convenient to obferve the fol-
lowing order of computation inftead of that before

4   prefcribed

prefcribed to be ufed when the altitudes are under 10 degrees.

1ft. Making ufe of the apparent altitudes of the Moon and ftar uncorrected, compute arches the firft and fecond by the directions contained in the rule of refraction.

2dly. Taking arch the fecond for arch A in the rule of parallax, compute the effect of parallax according to rule the fecond.

3dly. With arches the fifft and fecond compute the effect of refraction by rule the firft.

4thly, and laftly. Applying the two corrections of parallax and refraction duly, according to the rules, to the obferved diftance of the Moon from the ftar, you will have the true and correct diftance of the Moon from the ftar, cleared both of refraction and parallax.

A R U L E

For computing a fecond, but fmaller correction than the firft, neceffary to be applied to the obfervations of the diftance of the Moon from a ftar on account of parallax.

Call the principal effect of parallax, found by the preceding rule, the parallax in diftance ; and find the parallax anfwering to the Moon's altitude. Then to the conftant logarithm 0.941 add the logarithm of the fum of the parallax in altitude and the parallax in diftance, the logarithm of the difference of the fame parallaxes, and the cotangent of the obferved diftanse of the Moon from the ftar (corrected for refraction, and the principal effect of parallax), the fum, abating 13 from the index, is the

logarithm

logarithm of the number of feconds required, being the fecond correction of parallax; and is always to be added to the diftance of the Moon from the ftar, firft corrected for refraction, and the principal effect of parallax found above, in order to obtain the true diftance; unlefs the diftance exceeds 90 degrees, in which cafe it is to be fubftracted.

DEMONSTRATION.

Let L Fig. 6. reprefent the Moon's true place in the fphere, and r her apparent place as depreffed by parallax, S the place of the ftar, and L t a perpendicular let fall from the true place of the Moon L upon the great circle r S joining the ftar S and the apparent place of the Moon r; (all as in the four figures belonging to the preceding rule). Let L a be the arch of a parallel circle defcribed from the ftar S as a pole through the true place of the Moon L. S a terminated by the parallel circle L a, and not S t terminated by the perpendicular L t, as was fuppofed in the former demonftration, is equal to S L or the true diftance of the Moon from the ftar, which was therefore computed too fmall from the former rule by the little fpace $a t$. Let L T and a T be the equal tangents of the equal arches L S and a S in L and a, meeting in the radius C S, drawn from the centre of the fphere C and produced, in T. The fpace L $a t$, on account of its fmallnefs, may be looked upon as lying all in one plane namely L a T, and L a as the fmall arch of a circle defcribed from the point T as a centre with the line L T as a radius,

thro'

through L and a, Lt as the fine, and at as the verfe-fine of the arch La; and confequently at equal to the fquare of Lt divided by 2 L T. But, the triangle Lrt being right-angled in t, the fquare of Lt is equal to the difference of the fquares of Lr and rt, and confequently to the product of their fum and difference; that is to fay, $at = \dfrac{\overline{Lr+rt} \times \overline{Lr-rt}}{2\,LT}$ or (becaufe the tangent T L is equal to the fquare of the radius C S divided by the cotangent of L S) $= \overline{Lr+rt} \times \overline{Lr-rt}\,\dfrac{\times \text{ cotangent of L S}}{2 \times \text{ fquare of C S}}$. Now fuppofe the fpaces Lr, rt to be expreffed in minutes, which will be moft convenient in practice, then the radius of the fphere C S muft be taken equal to $3437\frac{3}{4}$, for fo many minutes are contained in an arch of a circle equal to its radius: and at will be $= \dfrac{\overline{Lr+rt} + \overline{Lr-rt} \times \text{ cotan. of L S}}{2 \times 3437\frac{3}{4} \times 3437\frac{3}{4}}$. But, the contangents of fimilar arches of circles of different radii being directly as the radii, therefore the cotangent of L S to the radius C S or $3437\frac{3}{4}$, is to the cotangent of the fame arch to 10000000000, which is the radius to which the logarithmic tables are adapted, its logarithm being 10; as $3437\frac{3}{4}$ to 10000000000. Therefore the cotangent of L S $=$ tabular cotangent of $\dfrac{\text{L S} \times 3437\frac{3}{4}}{10000000000}$, which, being fubftituted in the value of at above, gives at, expreffed in minutes $= \dfrac{\overline{Lr+rt} + \overline{Lr-rt} \times \text{ tabular cotangent L S}}{68755000000000}$; or, multiplying by 60, the value of at will come out in feconds

$$\text{conds} = \frac{\overline{Lr + rt} \times \overline{Lr - rt} \times \text{tabular cotangent of L S}}{1146000000000}.$$

The logarithm of the denominator 1146000000000 is 12,059, inftead of fubftracting which, when the operation is performed by logarithms, add 0,941 (its compliment to 13) to the value of the numerator found in logarithms, and fubftract 13 from the index: the remainder will be the value of *at* in feconds. Q. E. D.

A concife rule to find the diftance of the Moon from a zodiacal ftar, very nearly; the difference of the longitudes of the Moon and ftar, and the latitudes of both being given.

To the cofine of the difference of the longitudes add the cofine of the difference of the latitudes, if both of the fame denomination, or fum; if of contrary denominations, the fum of the two logarithms, abating 10 from the index, is the cofine of the approximate diftance. This gives the true diftance of the Moon from the Sun, being then nothing more than the common rule for finding the hypothenufe of a right-angled fpherical triangle from the two fides given. But in the cafe of a zodiacal ftar apply the following correction to the approximate diftance thus found.

To the conftant logarithm 5.3144 add the fine of the Moon's latitude, the fine of the ftar's latitude, the verfe-fine of the difference of longitude, and the cofecant of the approximate diftance; the fum of thefe 5 logarithms, abating 40 from the index, is the logarithm of a number of feconds, which fub-
ftracted

ſtraćted from the approximate diſtance, found before, if the latitudes of the moon and ſtar are of the ſame denomination, or added thereto, if they are of different denominations, gives the true diſtance of the Moon from the ſtar.

N. B. This rule, though only an approximation, is ſo very exaćt, that even, if the latitude of the Moon was 5°, and that of the ſtar 15°, the error would be only 10″; and if the latitude of the Moon be 5°, and that of the ſtar 10°, the error is only 4″ $\frac{1}{4}$; and if the latitudes be leſs, will be leſs in proportion as the ſquares of the ſines of the latitudes decreaſe.

DEMONSTRATION.

Let P [Fig. 7.] repreſent one of the poles of the eclip-and Q, R the places of the Moon and ſtar. From R let the arch of a great circle R D be drawn perpendicular to P Q. By ſpherics, the tangent of P D = tangent of P R × coſine of the angle R P D. And, by trigonometry, coſine of Q D or (Q P — P D) = coſ. Q P × coſ. P D + ſin. Q P × ſin. P D = coſ. Q P × coſ. P D + ſin. Q P × coſ. P D × tan. P D = coſ. P D × $\overline{\text{coſ. Q P + ſin. Q P} \times \text{tan. P D}}$ = coſ. P D × $\overline{\text{coſ. Q P} + \text{ſin. Q P} \times \text{tan. P R} \times \text{coſ. P}}$ ∴ coſ. Q D : coſ. P D :: coſ. Q P + ſin. Q P × tan. P R × coſ. P : 1. But, by ſpherics, coſ. Q D : coſ. P D :: coſ. R Q : coſ. P R ∴ coſ. R Q : coſ. P R :: coſ. Q P + ſin. Q P × tan. P R × coſ. P : 1. Whence coſ. R Q = coſ. P R × coſ. Q P + ſin. Q P × ſin. P R × coſ. P. Now, by trigonometry, coſ. (Q P — P R) = coſ.

$= \text{cof. } Q P \times \text{cof. } P R + \text{fin. } Q P \times \text{fin. } P R$; whence fin. $Q P \times$ fin. $P R = \text{cof. } (Q P - P R) -$ cof. $Q P \times$ cof. $P R$; which being fubftituted above, gives cof. $R Q = \text{cof. } (Q P - P R) \times \text{cof. } P - \text{cof.}$ $P R \times \text{cof. } Q P \times \text{cof. } P + \text{cof. } P R \times \text{cof. } Q P =$ cof. $(Q P - P R) \times \text{cof. } P + \text{verfe-fine } P \times \text{cof.}$ $P R \times \text{cof. } Q P$. Now put cof. $(Q P - P R) \times$ cof. $P = \text{cof. } G$, or the approximate diftance, then cof. $R Q - \text{cof. } G$, or (becaufe the difference of $R Q$ and G is but fmall) $\overline{G - R Q} \times \text{fin. } \left(\dfrac{G + R Q}{2}\right)$ $= \text{verfe-fine } P \times \text{cof. } P R \times \text{cof. } P Q$ nearly. Whence $R Q = G - \dfrac{\text{verfe-fine } P \times \text{cof. } P R \times \text{cof. } Q P}{\text{fin. } \frac{G + R Q}{2}}$

nearly $= G - \dfrac{v. \, f. \, P \times \text{cof. } P R \times \text{cof. } Q P}{\text{fin. } G}$ nearly. Q. E. D.

Note, the error of this formula arifes from taking $G = \dfrac{G + R Q}{2}$ by which means it will always give $R Q$ too great, nearly by the following quantity, $\frac{1}{2}$ $S q$ cof. $P R \times S q$ cof. $Q P \times$ cot. $G \times S q$ tan. $\frac{1}{2} G$. This comes to a maximum when G is $60°$, and is then $= \frac{1}{6} \sqrt{\frac{1}{3}} \times S q$ cof. $R P \times S q$ cof. $P Q$. If the latitudes of the Moon and ftar are both $5°$ it is $= 1''$. If the Moon's latitude be $5°$, and that of the ftar $10°$, it is $= 4'' \frac{1}{2}$; and if the latitude of the ftar be $15°$ it is $= 10''$.

XLIX.